Math. 102, Calculus II — Date: 01/06/2006
Final Exam. — Duration: 2 hours.

Answer all questions. Calculators and Mobile Phones are not allowed.

1. (4pts.)

   (a) Find the exact value of $\sin(\tan^{-1}(\frac{2}{3}))$.

   (b) Find the exact value of $\coth(\ln 2)$.

2. (4pts.) Let $f(x) = \int_0^x \tan(t)\sec^2(t)dt, 0 \le x < \pi/2$. Find

   (a) $f'(x)$

   (b) $f(\pi/4)$

   (c) $(f^{-1})'(\frac{1}{2})$

3. (12pts.) Find, if possible,

   (a) $\int \frac{\tan^{-1} x}{x^2} dx$

   (b) $\int_0^1 \frac{\ln x}{\sqrt{x}} dx$

   (c) $\int_1^2 \frac{x}{3x^2 - 6x + 4} dx$

4. (4pts.) A curve C is given parametrically by $x = (t^2+1)e^t$ and $y = (t^3 + 2t^2)e^t$.

   (a) Find the points $P(x, y)$ where the tangent is vertical.

   (b) Determine whether C is concave upwards or downwards at $t = 0$

5. (4pts.) Find the area of the surface obtained by rotating the curve $y = \sqrt{x+1}, 1 \le x \le 5$ about the x-axis.

6. (4pts.) Find the centroid (center of mass) of the region bounded by the curves $y = \sin x, y = 0, x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.

7. (4pts.) Find the length of the curve $r = 4\cos^2(\frac{\theta}{2})$.

8. (4pts.) Find the area enclosed by one loop of the curve $r = \sin 3\theta$.

June 2006, Math. 102 solutions

1. (a) Put $x = \tan^{-1}\frac{2}{3}$. Then $\tan x = 2/3$, (1pt) so $\sin x = \sin(\tan^{-1}(\frac{2}{3})) = 2/\sqrt{13}$ (1pt).

   (b) Put $y = \ln 2$. Then $e^y = 2, e^{-y} = 1/2$, (1pt) so $\coth y = \frac{e^y+e^{-y}}{e^y-e^{-y}} = 5/3$ (1pt).

2. (a) $f'(x) = \tan(x)\sec^2(x)$. (1pt)

   (b) $f(\pi/4) = \int_0^{\pi/4}\tan(t)\sec^2(t)dt = [\frac{1}{2}\tan^2 x]_0^{\pi/4} = 1/2$ (1.5pt)

   (c) $(f^{-1})'(\frac{1}{2}) = 1/f'(\pi/4) = 1/2$. (1.5pt)

3. (a) Set $I = \int \frac{\tan^{-1}x}{x^2}dx$ and put $u = \tan^{-1}x, dv = dx/x^2$. Then $v = -1/x$ and $I = -\frac{1}{x}\tan^{-1}x + \int\frac{dx}{x(1+x^2)}$(1pt) . Now, $\frac{1}{x(1+x^2)} = \frac{a}{x} + \frac{bx+c}{1+x^2} = \frac{1}{x} - \frac{x}{1+x^2}$(2pt). Thus $I = -\frac{1}{x}\tan^{-1}x + \ln|x| - \frac{1}{2}\ln(1+x^2) + C$ (1pt).

   (b) $\int\frac{\ln x}{\sqrt{x}}dx = \int \ln x d(2\sqrt{x}) = 2\sqrt{x}\ln x - 2\int x^{-1/2}dx = 2\sqrt{x}\ln x - 4\sqrt{x}$. Our integral becomes $\lim_{t\to 0^+}\int_t^1(\ln x/\sqrt{x})dx = \lim_{t\to 0^+}(-4+4\sqrt{t}-2\sqrt{t}\ln t)$. Now, $\lim_{t\to 0^+}\sqrt{t}\ln t = \lim_{t\to 0^+}\frac{\ln t}{t^{-1/2}}$
   (L'H rule) $= \lim_{t\to 0^+}\frac{1/t}{-\frac{1}{2}t^{-3/2}} = 0$. Thus $\int_0^1\frac{\ln x}{\sqrt{x}}dx = -4$

   (c) Set $J = \int\frac{x}{3x^2-6x+4}dx$. Since $3x^2 - 6x + 4 = 3[(x-1)^2 + \frac{1}{3}]$, we put $u = x - 1$(1pt). Then $J = \frac{1}{3}\int\frac{u+1}{u^2+\frac{1}{3}}du = \frac{1}{6}\int\frac{2u}{u^2+\frac{1}{3}}du + \frac{1}{3}\int\frac{du}{u^2+\frac{1}{3}}$(1pt)
   $= \frac{1}{6}\ln[(x-1)^2+\frac{1}{3}] + \frac{1}{\sqrt{3}}\tan^{-1}[\sqrt{3}(x-1)]$(1pt). Thus
   $\int_1^2\frac{x}{3x^2-6x+4}dx = [\frac{1}{6}\ln[(x-1)^2+\frac{1}{3}]+\frac{1}{\sqrt{3}}\tan^{-1}[\sqrt{3}(x-1)]]_1^2 = \frac{\ln 2}{3} + \frac{\pi}{3\sqrt{3}}$(1pt).

4. $dy/dx = \frac{dy/dt}{dx/dt}$ and $dy/dt = t(t+1)(t+4)e^t, dx/dt = (t+1)^2e^t$(1pt).

   (a) Thus $dy/dx = \frac{dy/dt}{dx/dt} = \frac{t(t+4)}{t+1}, t \neq -1$.

   Now $dx/dt = 0 \iff t = -1$ and $\lim_{t\to -1\pm} dy/dx = \lim_{t\to -1\pm}\frac{t(t+4)}{t+1} = \mp\infty$. Thus, the tangent is vertical only at $t = -1$(1pt).

   (b) $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{\frac{d}{dt}\frac{t(t+4)}{t+1}}{\frac{dx}{dt}} = \frac{t^2+2t+4}{(t+1)^4e^t} = 4$ when $t = 0$(1pt). Thus the curve is concave upwards(1pt).

5. We have $dy/dx = \frac{1}{2\sqrt{1+x}}$, so $y\sqrt{1+(dy/dx)^2} = y\sqrt{1+\frac{1}{4(x+1)}} = \sqrt{4x+5}$(2pt). Thus the surface area is $2\pi\int_1^5 y\sqrt{1+(dy/dx)^2}dx = 2\pi[\frac{1}{6}(4x+5)^{3/2}]_1^5 = \frac{98\pi}{3}$(2pt).

6. $\bar{x} = \frac{1}{A}\int_{\pi/4}^{3\pi/4} x\sin x dx, \bar{y} = \frac{1}{A}\int_{\pi/4}^{3\pi/4}\frac{1}{2}\sin^2 x dx$(1pt). Now,
   $A = \int_{\pi/4}^{3\pi/4}\sin x dx = -\cos x|_{\pi/4}^{3\pi/4} = \sqrt{2}$(1/2pt),
   $\int_{\pi/4}^{3\pi/4} x\sin x dx = [-x\cos x + \sin x]_{\pi/4}^{3\pi/4} = \frac{\pi}{\sqrt{2}}$(1pt),
   $\int_{\pi/4}^{3\pi/4}\frac{1}{2}\sin^2 x dx = \frac{1}{4}\int_{\pi/4}^{3\pi/4}(1-\cos 2x)dx = \frac{1}{4}[x-\frac{1}{2}\sin 2x]_{\pi/4}^{3\pi/4} = \frac{1}{4}(1+\pi/2)$(1/2pt).
   Thus $\bar{x} = \pi/2, \bar{y} = \frac{1}{4\sqrt{2}}(1+\pi/2)$(1pt).

7. Here $r = 2(1+\cos\theta)$, so $\sqrt{r^2 + (dr/d\theta)^2} = \sqrt{8(1+\cos\theta)} = \sqrt{16(\cos^2\frac{\theta}{2})} = 4\cos\frac{\theta}{2}$(2pt).

This cardioid is traced out as θ ranges from 0 to 2π, but as the curve is symmetric about the polar axis, the length of the curve is $2\int_0^{\pi} 4\cos\frac{\theta}{2}d\theta = 16[\sin\frac{\theta}{2}]_0^{\pi} = 16$(2pt).

8. The curve is at the origin when $\theta = 0$ and again when $3\theta = \pi$. Thus a loop is traced out as θ ranges from 0 to $\pi/3$(1pt). Thus the area of the loop is $\frac{1}{2}\int_0^{\pi/3} \sin^2 3\theta d\theta =$(1pt) $\frac{1}{4}\int_0^{\pi/3}(1-\cos 6\theta)d\theta = \frac{1}{4}[\theta - \frac{1}{6}\sin 6\theta]_0^{\pi/3} = \frac{\pi}{12}$(2pt).